Exactly solvable annealed random-bond Ising models in two dimensions

1994 J. Phys. A: Math. Gen. 27 4083

(http://iopscience.iop.org/0305-4470/27/12/016)

J. Phys. A: Math. Gen. 27 (1994) 4083–4095. Printed in the UK

Exactly solvable annealed random-bond Ising models in two dimensions

Makoto Idzumi†

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

Received 5 January 1994

Abstract. Annealed random-bond Ising models with frustration for the square lattice in two dimensions are considered. Three special models are solved exactly by mapping them to their dual models. Thermodynamic quantities are calculated analytically for the three models.

1. Introduction

To explore some realistic properties in condensed matter, introducing randomness would be crucial. Randomness often causes drastic changes in thermodynamics. Spin glass is a typical example in which randomness in the inter-spin interaction causes frustration in dominant configurations of spins. The best model of spin glass is considered to be a quenched random-bond spin models; but it seems hard to solve such a model with quenched randomness exactly. Alternatively we can consider annealed randomness which also causes frustration. We will consider an annealed random-bond Ising model and show that the thermodynamics of some special models can be analysed by exact solutions.

Usually an annealed random-bond Ising model is defined by a partition function

$$Z = \frac{1}{2^M}\sum_J\sum_\sigma \exp\left(-\beta\left(-\sum_{(\alpha,\beta)} J_{\alpha\beta}\sigma_\alpha\sigma_\beta\right)\right)$$

where $\beta \equiv 1/T$, $M = \#\{\text{bonds}\}$, and each bond interaction takes values $\pm J^0$ with equal probabilities. Models we consider are, however, slightly modified ones:

$$Z = \frac{1}{2^M}\sum_J\sum_\sigma \exp\left(-\beta\left(-\sum_{(\alpha,\beta)} J_{\alpha\beta}\sigma_\alpha\sigma_\beta + \sum_{(\alpha,\beta,\gamma,\delta)} gJ_{\alpha\beta}J_{\beta\gamma}J_{\gamma\delta}J_{\delta\alpha}\right)\right). \tag{1}$$

Here we assume the two-dimensional square lattice on a torus. The first sum is taken over all bonds (nearest-neighbour pairs) and the second is over all plaquettes (or unit squares). We can control the strength of the frustration by changing the real parameter g: as g becomes positive large the Ising spins tend to frustrate; as g becomes negative large the frustration gradually fades away. We show that the models are exactly solvable for three special values of g, which are:

(i) $g = 0$, the ordinary annealed model (the trivial case);

(ii) $g = +\infty$, which we call the *fully frustrated model*;

† Present address: Department of Mathematics (Education), Shimane University, Matsue, Shimane 690, Japan.

(iii) $g = -\infty$, which we call the *frustration-forbidden model.*
These three models are related to Ising models *without randomness* in an infinite magnetic field, in a purely imaginary magnetic field, $i\pi/2$, and in a zero field, respectively. A general connection between models (1) and Ising models in a magnetic field will be established in section 3. The zero-field Ising model was solved by Onsager [1] and the $i\pi/2$ field model was solved by McCoy and Wu [2]. Therefore we can solve models (1) in the three cases using their solutions. This is the content of this paper. Unfortunately we cannot solve any models for other values of g. This is related to the fact that the Ising model in a generic magnetic field has not yet been solved.

In this paper we summarize exact results for the three exactly solvable annealed random-bond Ising models. In section 2 we define the models. In section 3 we introduce *dual models* and establish the general connection mentioned above in a two-dimensional periodic square lattice. Explicit analytical expressions for the thermodynamic quantities for the three models are given in section 4 together with temperature-dependence figures.

2. Models

We limit ourselves to Ising models on the two-dimensional square lattice of N sites and M ($= 2N$) bonds with periodic boundary conditions. Let $\sigma_\alpha \in \{+1, -1\}$ be an Ising variable on a site α, and let $\tau_{\alpha\beta} \in \{+1, -1\}$ be a random variable attached to a bond (α, β). Let $J^0_{\alpha\beta} > 0$ be bond-dependent interaction constants, and let $g_{\alpha\beta\gamma\delta}$ be the plaquette-dependent frustration strength and we think of these as *parameters*; see figure 1(a). We consider annealed random-bond Ising models, defined by the partition functions

$$Z = \frac{1}{2^M} \sum_\tau \sum_\sigma \exp\left(\sum_{(\alpha,\beta)} L_{\alpha\beta}\tau_{\alpha\beta}\sigma_\alpha\sigma_\beta - \sum_{(\alpha,\beta,\gamma,\delta)} H^*_{\alpha\beta\gamma\delta}\tau_{\alpha\beta}\tau_{\beta\gamma}\tau_{\gamma\delta}\tau_{\delta\alpha} \right) \qquad (2)$$

where $L_{\alpha\beta} = J^0_{\alpha\beta}/T$, $H^*_{\alpha\beta\gamma\delta} = g_{\alpha\beta\gamma\delta} J^0_{\alpha\beta} J^0_{\beta\gamma} J^0_{\gamma\delta} J^0_{\delta\alpha}/T$, and T is the temperature. The annealing has already been performed: we have assumed that $\tau_{\alpha\beta}$ are independent random variables and that each of them has a common probability distribution $\mathrm{Prob}(\tau_{\alpha\beta} = +1) = \mathrm{Prob}(\tau_{\alpha\beta} = -1) = \frac{1}{2}$.

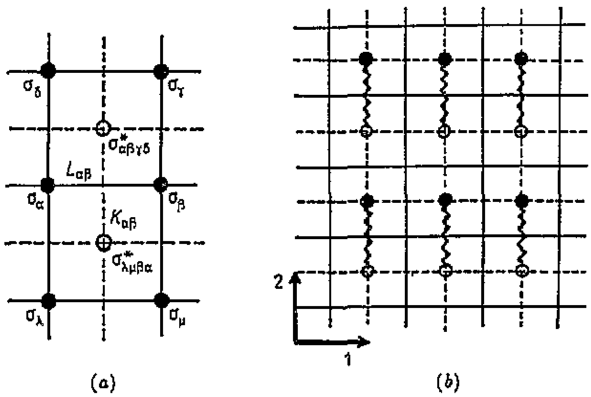


Figure 1. (a) The square lattice and its dual. (b) A set of bonds S (indicated by wavy lines); see (10).

Historically models (1) for $g < 0$ were introduced in lattice gauge theories and the ones for $g > 0$ were once proposed as effective models of Ising spin glasses (cf [3]).

It is easily seen that the interfacial free energies of the models are identically zero; the reason is quite simple: consider a model defined as in (2) but with the interaction constants $L_{n\mathbf{2},\mathbf{1}+n\mathbf{2}}$, $n = 1, 2, \ldots$, in a row of horizontal bonds are replaced by $-L_{n\mathbf{2},\mathbf{1}+n\mathbf{2}}$, where $\mathbf{1}$ and $\mathbf{2}$ are unit lattice vectors in the x and y directions, respectively, as indicated in figure 1(*b*). This row of horizontal bonds is considered as a *boundary*. Let Z_i be the partition function of the model. If in Z_i we negate all dummy variables $\tau_{n\mathbf{2},\mathbf{1}+n\mathbf{2}}$ in the row, we see that Z_i has the same form as the original Z. This means that the free energies for the two models are identical; hence the interfacial free energy of model (2) is zero.

We also note that a naive introduction of a magnetic field in models (2) does not result in any interesting magnetic properties. To see this add a new term $\sum h_\alpha \sigma_\alpha$ to the exponent in (2) and let Z_h be the partition function with this term, where we consider h_α to be a magnetic field. We see readily that

$$Z_h = \prod_\alpha \cosh h_\alpha \cdot Z$$

where Z is the partition function in zero field (2). Therefore, with respect to magnetic properties this model is identical to N free spins.

By similar reasoning it is easily seen that the correlations of σ variables are identically zero. Only mixed correlations among σ and τ variables can be non-trivial but we shall, however, not consider these in this paper.

3. The dual models

In this section we prove a general connection between the annealed random-bond Ising models (2) and Ising models in a magnetic field without randomness, which can be interpreted as a mapping from one to the other and *vice versa*.

Consider Ising models in a magnetic field on the *dual* lattice:

$$Z(H) = \sum_{\sigma^*} \exp\left(\sum_{(\alpha,\beta)} K_{\alpha\beta} \sigma^*_{\alpha\beta\gamma\delta} \sigma^*_{\lambda\mu\beta\alpha} + \sum_{(\alpha,\beta,\gamma,\delta)} H_{\alpha\beta\gamma\delta} \sigma^*_{\alpha\beta\gamma\delta} \right) \tag{3}$$

(see figure 1(*a*)).

Proposition. Provided that

$$e^{-2K_{\alpha\beta}} = \tanh L_{\alpha\beta} \qquad e^{-2H_{\alpha\beta\gamma\delta}} = -\tanh H^*_{\alpha\beta\gamma\delta} \tag{4}$$

the following identity holds

$$Z = 2^N \prod_{(\alpha,\beta)} e^{-K_{\alpha\beta}} \cosh L_{\alpha\beta} \prod_{(\alpha,\beta,\gamma,\delta)} e^{-H_{\alpha\beta\gamma\delta}} \cosh H^*_{\alpha\beta\gamma\delta} \cdot Z(H). \tag{5}$$

In this sense we call models (3) the *dual* models to (2).

The proof is elementary: we have to perform the high-temperature expansion of (2) and the low-temperature expansion of (3) (it will be helpful to use the standard polygon picture in interpreting the expansions); comparing both these expansions immediately yields (5).

We note that an extension of this identity, which is for a square lattice in two dimensions, to general lattices (including irregular ones) in arbitrary dimensions was established in [4].

Identity (5) gives us the following correspondence: models (2) for $g > 0$ (i.e. $H^* > 0$) correspond to models (3) with magnetic field $H_{\alpha\beta\gamma\delta} = H^{\mathrm{R}}_{\alpha\beta\gamma\delta} + \mathrm{i}\pi/2$, $H^{\mathrm{R}}_{\alpha\beta\gamma\delta}$ being real and positive (other choices for the imaginary part are possible), hence, $\mathrm{e}^{-2H^{\mathrm{R}}_{\alpha\beta\gamma\delta}} = \tanh H^*_{\alpha\beta\gamma\delta}$; models (2) for $g < 0$ (i.e. $H^* < 0$) correspond to models (3) with magnetic field $H_{\alpha\beta\gamma\delta} > 0$. In particular, a model (2) in the limit of $g \to +\infty$ corresponds to the Ising model in a pure imaginary magnetic field $\mathrm{i}\pi/2$ which was solved by McCoy and Wu [2], and model (2) in the limit of $g \to -\infty$ corresponds to the zero-field model solved by Onsager [1]. Note that in the limit $g \to +\infty$ only such configurations of τs that cause frustration are allowed, and that only ones that cause no frustration are allowed in the limit $g \to -\infty$.

Correlation functions for the dual models have an interesting interpretation when they are translated into the original models (2).

Proposition. Let $g \neq 0$ and let $\langle \cdots \rangle_H$ be the statistical average in the dual model (3). Then

$$Z_{\mathrm{d}} = Z \cdot \langle \sigma^*_{x_1} \cdots \sigma^*_{x_n} \rangle_H \tag{6}$$

where Z is the partition function (2), and Z_{d} is the partition function of a *disturbed* model defined by replacing plaquette constants g_{x_i} at x_i to $-g_{x_i}$ for all $i = 1, \ldots, n$ in (2).

We can think of the disturbed model as a model with *defects* at the plaquettes $x_1, \ldots, x_n$. The proof is easy (use $\sigma = \exp(-\mathrm{i}\pi\bar{\sigma}/2)$ in the RHS); so we omit it. From this the free energy shift caused by the defects is

$$\Delta_{\mathrm{d}} F \equiv F_{\mathrm{d}} - F = -T \log(Z_{\mathrm{d}}/Z) = -T \log\langle \sigma^*_{x_1} \cdots \sigma^*_{x_n} \rangle_H . \tag{7}$$

In the special cases $H = \mathrm{i}\pi/2$ and $H = 0$ the correlations $\langle \sigma^*_{x_1} \cdots \sigma^*_{x_n} \rangle_H$ can be calculated analytically [2, 5]; hence we can calculate the $\Delta_{\mathrm{d}} F$ for the fully frustrated model and the frustration-forbidden model, respectively. This will be done in next section for two-point functions ($n = 2$) with $|x_2 - x_1| \gg 1$.

4. Exactly solvable models

In this section we give explicit analytical expressions for several thermodynamic quantities for the three solvable models, which are the free energy, the internal energy, the specific heat, and the entropy

$$-\frac{f}{T} \equiv \frac{1}{N} \log Z \qquad u \equiv -\frac{\partial}{\partial\beta} \frac{1}{N} \log Z \qquad c \equiv \frac{\mathrm{d}u}{\mathrm{d}T} \qquad s \equiv -\frac{(f-u)}{T}$$

and the free energy shift (7). In the cases $g = \pm\infty$ final expressions will be given for the homogeneous models in which $L_{\alpha\beta} = L$ for all bonds.

4.1. *The annealed model without constraint* ($g = 0$)

First consider the ordinary annealed model without constraint, $g = 0$. The partition function is computed as (perform the summation over τ configurations first)

$$Z_0 = 2^{-M} \sum_\tau \sum_\sigma \exp \sum_{(\alpha,\beta)} L_{\alpha\beta}\tau_{\alpha\beta}\sigma_\alpha\sigma_\beta = 2^N \prod_{(\alpha,\beta)} \cosh L_{\alpha\beta}. \tag{8}$$

We have written Z_0 to indicate that it is the partition function for $g = 0$. The free energy, the internal energy, the specific heat and the entropy are:

$$\begin{aligned}
-f_0/T &= \log 2 + \frac{1}{N}\sum_{(\alpha,\beta)} \log\cosh L_{\alpha\beta} \\
u_0 &= -\frac{1}{N}\sum_{(\alpha,\beta)} J^0_{\alpha\beta} \tanh L_{\alpha\beta} \\
c_0 &= \frac{1}{N}\sum_{(\alpha,\beta)} \frac{L^2_{\alpha\beta}}{(\cosh L_{\alpha\beta})^2} \\
s_0 &= \log 2 + \frac{1}{N}\sum_{(\alpha,\beta)} (\log\cosh L_{\alpha\beta} - L_{\alpha\beta}\tanh L_{\alpha\beta})
\end{aligned} \tag{9}$$

respectively.

In figure 2 their temperature dependences are shown for the homogeneous case $L_{\alpha\beta} = L$. Leading terms near $T = 0$ are $f_0 \approx -2 + T\log 2$, $u_0 \approx -2$, $c_0 \approx (8/T^2)\exp(-2/T)$ and $s_0 \approx -\log 2 + (4/T)\exp(-2/T)$; behaviours at high temperature are $f_0 \approx -T\log 2$, $u_0 \approx -2/T$, $c_0 \approx 2/T^2$ and $s_0 \approx \log 2$.

Negative entropy appears at low temperature because of our normalization in (8) which is artificial; if we count the degrees of freedom of the τ variables correctly as $\tilde{Z}_0 = \sum_\tau \sum_\sigma \exp \sum L\tau\sigma\sigma$, then the entropy becomes positive even at $T = 0$ ($\log 2$ at $T = 0$) and the free energy becomes a decreasing function near $T = 0$.

4.2. *The fully frustrated model* ($g = +\infty$)

Now we consider the fully frustrated model $g = +\infty$. We first calculate the partition function for the corresponding dual model, (3) with $H = \mathrm{i}\pi/2$, and later interpret the results as those for the fully frustrated model $g = +\infty$. The computation is described in some detail.

Let $H_{\alpha\beta\gamma\delta} = \mathrm{i}\pi/2$ in (3). The high-temperature expansion yields

$$Z(\mathrm{i}\pi/2) = 2^N \prod_{(\alpha,\beta)} \cosh K_{\alpha\beta} \cdot G$$

where

$$\begin{aligned}
G &= \frac{\mathrm{i}^N}{2^N} \sum_{\sigma^*} \prod_{(\alpha,\beta)} (1 + \sigma^*_{\alpha\beta\gamma\delta}\sigma^*_{\lambda\mu\beta\alpha} \tanh K_{\alpha\beta}) \prod_{(\alpha,\beta,\gamma,\delta)} \sigma^*_{\alpha\beta\gamma\delta} \\
&= \frac{\mathrm{i}^N}{2^N} \sum_{\sigma^*} \prod_{(\alpha,\beta)} (1 + \sigma^*_{\alpha\beta\gamma\delta}\sigma^*_{\lambda\mu\beta\alpha} \tanh K_{\alpha\beta}) \prod_{(\alpha,\beta)\in S} (\sigma^*_{\alpha\beta\gamma\delta}\sigma^*_{\lambda\mu\beta\alpha}).
\end{aligned}$$

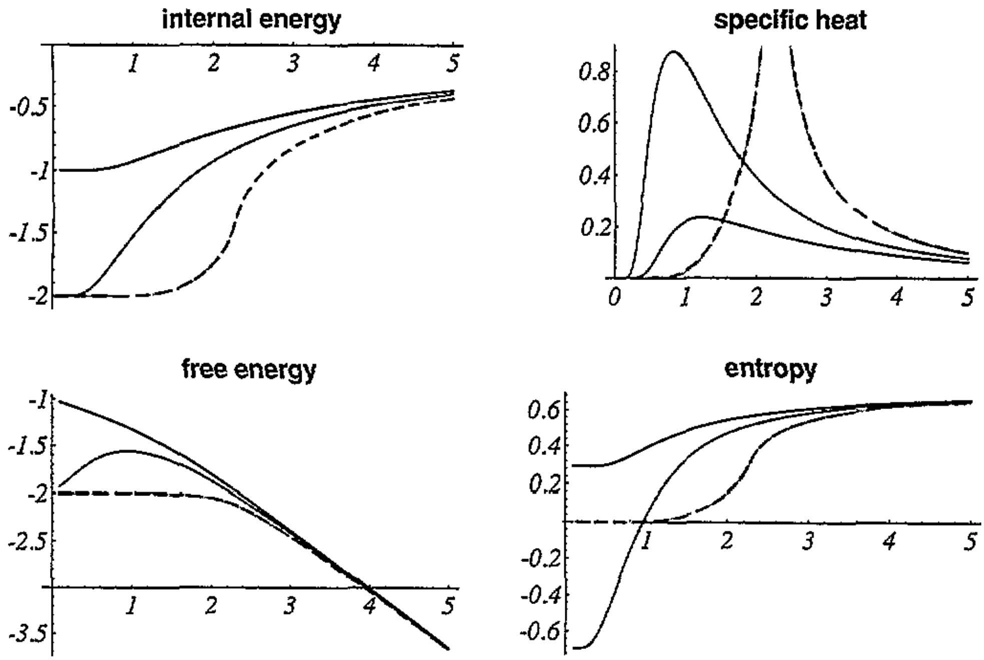


Figure 2. The internal energy u, the specific heat c, the free energy f and the entropy s are drawn as functions of temperature T for the fully frustrated model $g = +\infty$ (thick lines); the annealed model $g = 0$ (medium lines); the frustration-forbidden model $g = -\infty$ (light lines), respectively (unit of energy is J^0). The partition functions are normalized in such a way that $Z \to 2^N$ in the non-interaction limit.

Here S is a set of vertical bonds indicated by wavy lines in figure 1(b); $\#S = N/2$. In the following we assume that N is a multiple of 4. Rewriting the first line as the second is a key step [2]: the latter is exactly the N-point correlation function of the zero-field model which can be computed by the method developed by Montroll *et al* [5]. We can write

$$G = \frac{1}{2^N} \sum_{\sigma^*} \prod_{(\alpha,\beta)\in S} (1 + \sigma^*_{\alpha\beta\gamma\delta}\sigma^*_{\lambda\mu\beta\alpha} \coth K_{\alpha\beta}) \prod{}' (1 + \sigma^*_{\alpha\beta\gamma\delta}\sigma^*_{\lambda\mu\beta\alpha} \tanh K_{\alpha\beta})$$
$$= \operatorname{Pf} A \tag{10}$$

where $\prod'$ means the product over bonds which do not belong to S, and $\operatorname{Pf} A$ is a Pfaffian of an antisymmetric matrix A. The A is a $4N \times 4N$ matrix defined as in appendix A.1 except that $z_{\alpha-2,\alpha}$ for $\alpha \in$ (a set of sites indicated by $\bullet$ in figure 1(b)) and $z_{\alpha,\alpha+2}$ for $\alpha \in$ (a set of sites indicated by $\circ$ in figure 1(b)) are replaced by their inverses; here $z_{\alpha\beta} = \tanh K_{\alpha\beta}$; and **1** and **2** are unit lattice vectors in the x and y directions, respectively, as indicated in figure 1(b). (The description here would be clear enough if we are familiar with the Pfaffian method of Montroll *et al* [5].)

To proceed further we set $z_{\alpha\beta} = z_1 = \tanh K_1$ for horizontal bonds and $z_{\alpha\beta} = z_2 = \tanh K_2$ for vertical bonds. Then we can evaluate the Pfaffian using $(\operatorname{Pf} A)^2 = \det A$ as in [5] and [2]: noting that the block matrix A is invariant under translations by one unit in the x direction and by two units in the y direction, the Fourier transform converts A into block-diagonal form in which each block is an 8×8 matrix; the determinant is therefore calculable by hand. The result is [2]

$$\log\det A = \frac{N}{2}\frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi} \mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\log\det\hat{A}(\varphi_1,\varphi_2)$$

where

$$\det\hat{A}(\varphi_1,\varphi_2) = 4z_2^{-2}\bigg\{(1+z_2^4)z_1^2 + (1+z_1^4)z_2^2 - (1-z_2^2)^2 z_1^2(\cos\varphi_2)^2 - (1-z_1^2)^2 z_2^2\left(\cos\frac{\varphi_1}{2}\right)^2\bigg\}$$

and the φ-integrations appear instead of summations because we have already taken the thermodynamic limit. It gives the free energy of the fully frustrated model (the model (2) in the limit $g \to +\infty$, i.e. $H^* \to +\infty$):

$$\frac{1}{N}\log[Z/(\cosh H^*)^N]_{H^*=+\infty} = \tfrac{1}{4}\log 2 + L_1 + L_2 + \frac{1}{4}\frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi}\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\log\big\{(1+z_2^2)^2z_1^2 + (1+z_1^2)^2z_2^2 - (1-z_2^2)^2z_1^2\cos\varphi_2 - (1-z_1^2)^2z_2^2\cos\varphi_1\big\}$$

where $z_i = \mathrm{e}^{-2L_i} = \tanh K_i$ $(i = 1, 2)$. We re-define

$$Z_{+\infty} \equiv [Z/(\cosh H^*)^N]_{H^*=+\infty}$$

and regard this as the partition function of the model; the normalization has been chosen in such a way that if we set $z_1 = z_2 = 1$ (i.e. $L_1 = L_2 = 0$) we get $(1/N)\log Z_{+\infty} = \log 2$.

In order to avoid inessential complexity we further limit ourselves to the isotropic model:

$$L_1 = L_2 \equiv L \equiv J^0/T \qquad z_1 = z_2 \equiv z \equiv \mathrm{e}^{-2L}.$$

Then

$$\frac{1}{N}\log Z_{+\infty} = \tfrac{1}{2}\log 2\sinh 2L + \frac{1}{(2\pi)^2}\iint_0^{\pi}\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\log\{4(\coth 2L)^2 - 2(\cos\varphi_1+\cos\varphi_2)\}. \tag{11}$$

The standard manipulation yields (as in [1] and [6]; see appendix A.2)

$$-\frac{f}{T} \equiv \frac{1}{N}\log Z_{+\infty} = \log 2 + \tfrac{1}{2}\log\cosh 2L + \frac{1}{4\pi}\int_0^{\pi}\mathrm{d}\phi\,\log\left[\frac{1}{2}\left(1+\sqrt{1-k^2\sin^2\phi}\right)\right] \tag{12}$$

where $k \equiv (\tanh 2L)^2$, $0 < k < 1$.

Differentiating (12) we get analytical expressions for the internal energy and the specific heat:

$$u = -J^0k^{-1/2}\left(1 - \frac{2}{\pi}(1-k)K(k)\right) \tag{13}$$

$$c = \frac{4}{\pi}\left(\frac{J^0}{T}\right)^2\frac{1-k}{k}\left\{-\frac{\pi}{2} + (3-k)K(k) - \frac{2E(k)}{1+k}\right\} \tag{14}$$

where $K(k)$ and $E(k)$ are the complete elliptic integral of the first and the second kind, of modulus k:

$$K(k)=\int_0^{\pi/2}\frac{\mathrm{d}\phi}{(1-k^2\sin^2\phi)^{1/2}}\qquad E(k)=\int_0^{\pi/2}(1-k^2\sin^2\phi)^{1/2}\,\mathrm{d}\phi.$$

We have used a simple identity:

$$\frac{\sin^2\phi}{\Delta(1+\Delta)}=\frac{1}{k^2}\left(\frac{1}{\Delta}-1\right)$$

where $\Delta\equiv(1-k^2\sin^2\phi)^{1/2}$ and

$$k\frac{\mathrm{d}K(k)}{\mathrm{d}k}=\frac{E(k)}{1-k^2}-K(k).$$

The elliptic integral $K(k)$ has a singularity at $k=1$: in a neighbourhood of which

$$K(k)=\log\frac{4}{k'}+\frac{1}{2}\left[\log\frac{4}{k'}+1\right]k'^2+\cdots$$

$$E(k)=1+\frac{1}{2}\left[\log\frac{4}{k'}-\frac{1}{2}\right]k'^2+\cdots$$

where $k'\equiv\sqrt{1-k^2}$ (see, e.g., [7], formulae 900.05 and 900.10). In the Onsager's solution [1] this singularity corresponds to the critical point at which the second-order phase transition occurs. In the fully frustrated model we are analysing, the singular point $k\equiv(\tanh 2L)^2=1$ ($L=J^0/T$) corresponds to zero temperature; and hence the phase transition can occur at $T=0$.

In figure 2 we show the internal energy, the specific heat, the free energy and the entropy as functions of T/J^0 (the integration in (12) was performed numerically). Leading terms in the low-/high-temperature limit are obtained for (13) and (14); for $T\to 0^+$:

$$u=-1+\frac{16}{\pi}\frac{1}{T}\mathrm{e}^{-4/T}-\left(2-\frac{4}{\pi}\log 2\right)\mathrm{e}^{-4/T}+\cdots$$

$$c=\frac{64}{\pi}\frac{1}{T^3}\mathrm{e}^{-4/T}+\left(-\frac{16}{\pi}(1-\log 2)-8\right)\frac{1}{T^2}\mathrm{e}^{-4/T}+\cdots$$

and for $T\to+\infty$:

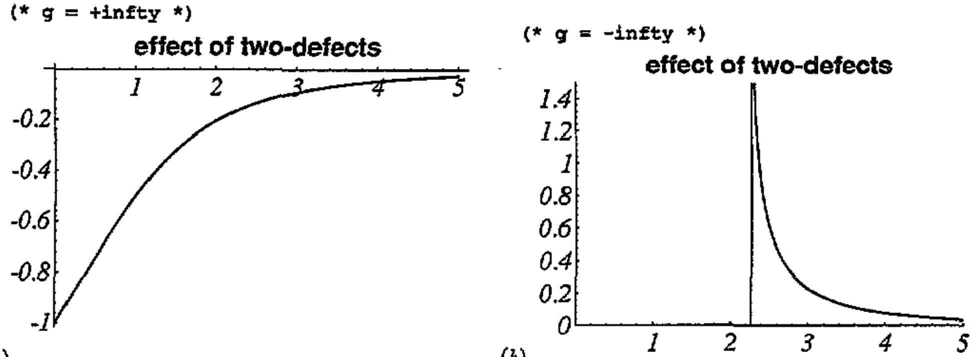
$$u=-\frac{2}{T}+\frac{14}{3}\frac{1}{T^3}-\frac{304}{15}\frac{1}{T^5}+\cdots\qquad c=\frac{2}{T^2}-\frac{14}{T^4}+\frac{304}{3}\frac{1}{T^6}-\cdots.$$

Here we have set $J^0\equiv 1$.

The internal energy is -1 at $T=0$ because of the constraint on configurations. The entropy remains positive since there are a huge number of minimum energy states.

The correlation $\langle\sigma_0^*\sigma_x^*\rangle_H$ for large $|x|$ was computed by McCoy and Wu [2] which would be identified with the square of the spontaneous magnetization of the dual model: (3) with $H=\mathrm{i}\pi/2$. Their result reads (equation (4.41) in [2]):

$$\langle\sigma_0^*\sigma_x^*\rangle_H=\left[\frac{(\cosh 2L)^4}{\cosh 4L}\right]^{1/4}$$



(*a*) (*b*)

Figure 3. The free energy shifts caused by two defects for the fully frustrated model and the frustration-forbidden model.

for *all* temperature (note that the critical temperature of the McCoy–Wu model [2] is $+\infty$) and then we obtain the free energy shift:

$$\begin{aligned}\Delta_{\mathrm{d}}F &\equiv F_{\mathrm{d}} - F = -T\log(Z_{\mathrm{d}}/Z) \\ &= -\frac{1}{4}T\log\frac{(\cosh 2L)^4}{\cosh 4L}\end{aligned} \tag{15}$$

for all temperature ($L = J^0/T$); see figure 3. Defects lower the free energy because they allow local energy-minimum configurations around them.

4.3. *The frustration-forbidden model* ($g = -\infty$)

The frustration-forbidden model $g = -\infty$ corresponds to the zero-field Ising model. The solution to the latter by Onsager [1] is famous. We only quote final expressions.

The partition function in the thermodynamic limit for the model $L_{\alpha\beta} = L_1, L_2$ for vertical, horizontal bonds, respectively, is

$$\begin{aligned}\frac{1}{N}\log[Z/(\cosh H^*)^N]_{H^*=-\infty} = L_1 + L_2 + \frac{1}{2}\frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi} \mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\log\big\{(1+z_1^2)(1+z_2^2) \\ - 2(1-z_2^2)z_1\cos\varphi_2 - 2(1-z_1^2)z_2\cos\varphi_1\big\}\end{aligned}$$

where $z_i = \mathrm{e}^{-2L_i} = \tanh K_i$ $(i = 1, 2)$. We re-define

$$Z_{-\infty} \equiv [Z/(\cosh H^*)^N]_{H^*=-\infty}$$

and regard this as the partition function of the model; the normalization has been chosen in such a way that if we set $z_1 = z_2 = 1$ we get $(1/N)\log Z_{-\infty} = \log 2$.

We further specialize the model to

$$L_1 = L_2 \equiv L \qquad z_1 = z_2 \equiv z \equiv \mathrm{e}^{-2L}.$$

Then

$$\begin{aligned}\frac{1}{N}\log Z_{-\infty} &= \tfrac{1}{2}\log 2\sinh 2L \\ &\quad + \frac{2}{(2\pi)^2}\iint_{0}^{\pi} \mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\log\left\{\frac{2(\cosh 2L)^2}{\sinh 2L} - 2(\cos\varphi_1 + \cos\varphi_2)\right\} &(16) \\ &= \log 2 + \log\cosh 2L + \frac{1}{2\pi}\int_0^{\pi}\mathrm{d}\phi\,\log\left[\frac{1}{2}\left(1+\sqrt{1-k^2\sin^2\phi}\right)\right] &(17)\end{aligned}$$

where $k \equiv 2\sinh 2L/(\cosh 2L)^2$, $0 < k < 1$.

The internal energy and the specific heat are

$$u = -J^0\sqrt{\frac{2}{1+k'}}\left(1+\frac{2}{\pi}k'K(k)\right) \tag{18}$$

$$c = \frac{2}{\pi}\left(\frac{J^0}{T}\right)^2\frac{2}{1+k'}\left\{2K(k)-2E(k)-(1-k')\left[\frac{\pi}{2}+k'K(k)\right]\right\} \tag{19}$$

where $k' \equiv 2(\tanh 2L)^2 - 1$.

A singularity occurs at $k = 1$, which corresponds to the critical temperature

$$T_c/J^0 = \frac{2}{\log(1/\sqrt{2}-1)} = 2.269\,185\ldots.$$

Hence the phase transition occurs at $T = T_c$.

In figure 2 we show the internal energy, the specific heat, the free energy and the entropy as functions of T/J^0 (the integration in (17) was performed numerically). The results show that there is an order at low temperature; there are many ordered states according to many τ configurations.

The correlation $\langle\sigma_0^*\sigma_x^*\rangle_H$ for large $|x|$, which would be identified with the square of the spontaneous magnetization of the dual model (3) with $H = 0$, is (see [5]):

$$\langle\sigma_0^*\sigma_x^*\rangle_H = \begin{cases} [1-(\sinh 2L)^4]^{1/4} & \text{for } T > T_c \\ 0 & \text{for } T < T_c. \end{cases}$$

This gives the free energy shift:

$$\Delta_d F = \begin{cases} -\frac{1}{4}T\log[1-(\sinh 2L)^4] & \text{for } T > T_c \\ +\infty & \text{for } T < T_c. \end{cases} \tag{20}$$

See figure 3. The total free energy increases divergently if defects are introduced (at low temperature) because the defects destroy the ordered configurations. A word of caution: infinity appears in equation (20) because, by definition, $\Delta_d F$ is a shift of total free energy; a shift of per-site free energy tends to zero. More precisely, by [8], we know that it grows infinitely as

$$\Delta_d F = \tfrac{1}{2}T(\log|x| + 2|x|\log(e^{2L}\tanh L)) + \cdots \qquad T < T_c$$

as $|x| \to +\infty$ ($|x|$ is the distance between the two defects).

5. Summary

We have considered annealed random-bond Ising models with frustration (2) [or (1)] on the two-dimensional square lattice with periodic boundary conditions. In section 3 we established a general connection between the annealed random-bond Ising models with frustration (2) and Ising models in a magnetic field (3). Note that the various constants $L_{\alpha\beta}$, $H^*_{\alpha\beta\gamma\delta}$, etc, are not necessarily constrained to real values in this connection (for a generalization see [4]). We saw that three special models are exactly solvable: these are the trivial model ($g = 0$), the fully frustrated model ($g = +\infty$), and the frustration-forbidden model ($g = -\infty$). In section 4 we gave exact analytical expressions for thermodynamic quantities for the three solvable models. It was seen that phase transitions do not occur at finite temperature for the models $g = 0, +\infty$, and that the second-order phase transition occurs for the $g = -\infty$ model.

Finally, it would be of great interest to draw a T against g phase diagram for the model (1). The analytical approach in this direction does not, however, seem easy and was beyond the scope of this paper.

Appendix

We assemble here the standard techniques for the Ising problem. The derivation of equation (10) in the text is based on the Pfaffian method of Montroll *et al* [5] (section A.1). Section A.2 is devoted to a technique which is used from (11) to (12), and (16) to (17) in the text (cf [1] and [6]).

A.1. Pfaffian representation of the partition function

In the Ising problem we often encounter the following expression:

$$G = \frac{1}{2^N} \sum_{\sigma^*} \prod_{(\alpha,\beta)} (1 + z_{\alpha\beta} \sigma^*_{\alpha\beta\gamma\delta} \sigma^*_{\lambda\mu\beta\alpha})$$

where the product is over all bonds. We have formulated the problem using a dual lattice as in the text. If the lattice under consideration is a two-dimensional square lattice we can write

$$G = \operatorname{Pf} A$$

where $\operatorname{Pf} A$ is a Pfaffian of an anti-symmetric matrix A; the A is a $4N \times 4N$ matrix defined by

$$A = \left[A(\alpha;\beta) \middle| \begin{matrix} \alpha \downarrow \\ \beta \rightarrow \end{matrix} \right] \qquad A(\alpha;\alpha) = \begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

$$A(\alpha;\alpha+1) = \begin{bmatrix} 0 & z_{\alpha,\alpha+1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A(\alpha;\alpha-1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -z_{\alpha-1,\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A(\alpha;\alpha+2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_{\alpha,\alpha+2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A(\alpha;\alpha-2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -z_{\alpha-2,\alpha} & 0 \end{bmatrix}.$$

Here $\mathbf{1}$ and $\mathbf{2}$ are unit lattice vectors in the x and y directions, respectively, as indicated in figure 1(b). Four columns (or rows) of the matrices correspond to sites R, L, U, D of the decorated lattice, as shown in figure A1 (standard notation; see [5]).

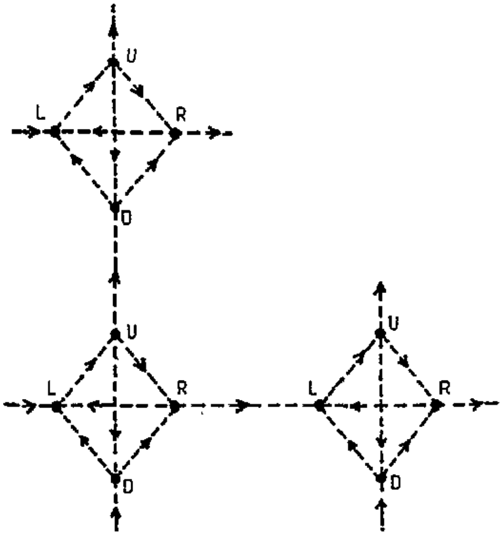


Figure A1. The decorated lattice obtained by giving structure to the lattice sites (of the dual lattice).

A.2. Reduction of a double integral to a single one

Here we describe a way to reduce double integrals which are often encountered in calculating the Ising partition functions.

$$\frac{1}{(2\pi)^2}\int\!\!\int_0^{\pi} \mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\log\{2D-2(\cos\varphi_1+\cos\varphi_2)\}$$

where D is a constant. We follow the standard manipulation (cf [1] or [6]). Changing integral variables to $\phi_1=(\varphi_1+\varphi_2)/2$, $\phi_2=\varphi_1-\varphi_2$ and the region of integration to $0<\phi_1$, $\phi_2<\pi$ yield

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int\!\!\int_0^{\pi} \mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\, m\log\{2D-2(\cos\varphi_1+\cos\varphi_2)\}\\
&\quad=\frac{1}{(2\pi)^2}\int\!\!\int_0^{\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\log\{2D-4\cos\phi_1\cos 2\phi_2\}\\
&\quad=\frac{1}{(2\pi)^2}\int\!\!\int_0^{\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\log\{2D-4\cos\phi_1\cos\phi_2\}\\
&\quad=\frac{1}{(2\pi)^2}\int\!\!\int_0^{\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\log(2\cos\phi_1)\\
&\qquad+\frac{1}{(2\pi)^2}\int\!\!\int_0^{\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\log\left(\frac{D}{\cos\phi_1}-2\cos\phi_2\right)\\
&\quad=\frac{1}{4\pi}\int_0^{\pi}\mathrm{d}\phi_1\,\log(2\cos\phi_1)+\frac{1}{4\pi}\int_0^{\pi}\mathrm{d}\phi_1\,\cosh^{-1}\left(\frac{D}{2\cos\phi_1}\right).
\end{aligned}$$

We have used an identity:

$$|x|=\frac{1}{\pi}\int_0^{\pi}\mathrm{d}t\,\log(2\cosh x-2\cos t).$$

Since $\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$ we can write the integrand in the second term by log. Finally

$$= \tfrac{1}{4}\log(2D) + \frac{1}{4\pi}\int_0^\pi \mathrm{d}\phi_1 \, \log\left[\frac{1}{2}\left(1 + \sqrt{1 - k^2 \sin^2 \phi_1}\right)\right]$$

where $k \equiv 2/D$.

References

[1] Onsager L 1944 *Phys. Rev.* **65** 117

[2] McCoy B M and Wu T T 1967 *Phys. Rev.* **155** 438

[3] Toulouse G and Vannimenus J 1980 *Phys. Rep.* **67** 47

[4] Idzumi M 1993 *J. Phys. A: Math. Gen.* **26** L919

[5] Montroll E W, Potts R B and Ward J C 1963 *J. Math. Phys.* **4** 308

[6] Huang K 1963 *Statistical Mechanics* (Chichester: Wiley) section 17.3

[7] Byrd P F and Friedman M D 1954 *Handbook of Elliptic Integrals for Engineers and Physicists* (Berlin: Springer)

[8] Wu T T 1966 *Phys. Rev.* **149** 380